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Optimum Filtering Required for Broadband Detection of Gaussian Processes

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Preface

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The derivation for the optimum filter required for broadband detection of Gaussian processes is presented. The performance criteria that is optimized is probability of detection for a fixed false alarm rate. It is shown by example that the optimum filter out performs the widely used Eckart filter and other filters.

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LIST OF SYMBOLS

t	Time
$x(t)$	Input waveform
$s(t)$	Input signal
$n(t)$	Input noise
M	Number of independent samples
T	Observation time
f	Frequency
$H(f)$	Filter transfer function
z	Decision variable
Λ	Threshold
$H_o(f)$	Optimum filter
S	Signal Power
N	Noise Power
Φ	Cumulative Gaussian distribution; Eq. 3
Φ^{-1}	Inverse Φ function
P_D	Probability of Detection
P_F	Probability of False Alarm
B	Bandwidth of Signal
$G_S(f)$	Signal Power Spectrum
$G_N(f)$	Noise Power Spectrum
Q	Measure of performance
$F(\ )$	Function of ( )
$E[\ ]$	Expected value
$Var[\ ]$	Variance
σ	Standard deviation
R	$G_S(f)/G_N(f)$; continuous or discrete
$\tilde{R}$	Estimate of R
$\underline{\tilde{R}}$	Collection of $\tilde{R}_i$

OPTIMUM FILTERING REQUIRED FOR BROADBAND DETECTION OF RANDOM PROCESSES

INTRODUCTION

It has long been believed that optimum filtering for broadband detection consists of using an Eckart filter. The Eckart filter optimizes the deflection criteria; however, it does not optimize the criteria of usual interest, which is maximum probability of detection (P_D) for a fixed false alarm rate.

In this report, we present the derivation of the optimum filter that maximizes P_D for a fixed false alarm rate. We show by example that the optimum filter out performs the Eckart filter and other filters. Additionally, we present a method of estimating performance when the input spectra are unknown and must be estimated.

PROBLEM DEFINITION

The configuration of interest is depicted in figure 1. The input x(t) is composed of stationary, zero-mean, Gaussian signal, s(t), and noise, n(t), or noise n(t) alone. The sampler and summer effectively accumulates M statistically independent samples during an observation time of T seconds.

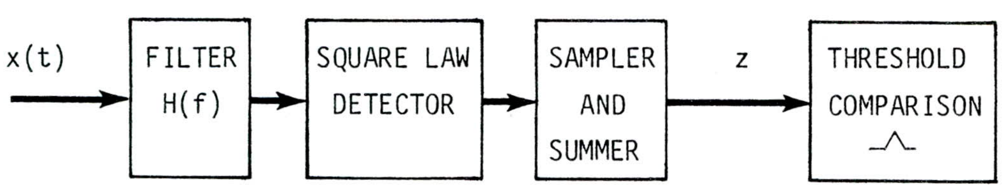

Figure 1. Processing Configuration

If the random variable z exceeds the threshold Λ when signal is present, a detection occurs. If z exceeds the threshold when signal is absent, a false alarm occurs. The problem is to find the optimum filter, $H_0(f)$, that maximizes P_D, for a given probability of false alarm, P_F, and a given observation time, T.

PROBLEM SOLUTION

We first define the signal-to-noise ratio (S/N) at the input of the square law detector in figure 1 that is required to detect a signal with given P_D and P_F, i.e.,

$$S/N \approx \frac{\Phi^{-1}(P_D) - \Phi^{-1}(P_F)}{\sqrt{M} - \Phi^{-1}(P_D)} , \tag{1}$$

$$M \approx BT, \tag{2}$$

where

$$\Phi(x) \equiv \int_{-\infty}^{x} dt(2\pi)^{-1/2} \exp(-t^2/2) \quad . \tag{3}$$

The derivation for the above is given by reference 1. The filter output S/N and bandwidth (B) are given by

$$S/N \equiv \frac{\int_0^\infty df \; |H(f)|^2 \; G_S(f)}{\int_0^\infty df \; |H(f)|^2 \; G_N(f)} , \tag{4}$$

and

$$B \equiv \frac{\left[\int_0^\infty df \; |H(f)|^2 \; G_S(f)\right]^2}{\int_0^\infty df \; |H(f)|^4 \; G_S(f)^2} \quad . \tag{5}$$

Equation (1) was derived by approximating the random variable z as Gaussian; this is a good approximation for $M \gg 1$.

Substituting (2) into (1) and rearranging, we obtain

$$P_D \approx \Phi\left[\frac{S/N \sqrt{BT} + \Phi^{-1} (P_F)}{1 + S/N}\right]. \tag{6}$$

Since Φ, as given by (3), is a monotonically increasing function, P_D can be maximized by maximizing its argument

$$\frac{S/N\sqrt{BT} + \Phi^{-1}(P_F)}{1 + S/N} \quad . \tag{7}$$

For the filter output $S/N << 1$, as is usually the case, and recognizing that $\Phi^{-1}(P_F)$ and T are constants, P_D can be maximized by maximizing Q, where

$$Q \equiv (S/N)^2\, B \quad . \tag{8}$$

Appendix A provides the derivation for the optimum filter $|H_o(f)|^2$ that maximizes Q. $|H_o(f)|^2$ is given by the following equation:

$$|H_o(f)|^2 = \frac{1}{R^2(f)\, G_N(f)} \left[R(f) - \frac{3I_1 - (9I_1{}^2 - 8I_oI_2)^{1/2}}{4I_o} \right] \tag{9}$$

where

$$R(f) = G_S(f)/G_N(f),$$

$$I_n \equiv \int_{W_o} df\, R^{n-2}(f).$$

W_o is the largest region where

$$9I_1^2 \geq 8I_oI_2.$$

Substituting (8) into (6) and utilizing $S/N << 1$, we have

$$P_D \approx \Phi\left[(QT)^{1/2} + \Phi^{-1}(P_F)\right] \quad . \tag{10}$$

COMPARISON WITH OTHER FILTERS

Other filters are presently in use for broadband processing. These include the ideal passband filter, the Eckart filter, and the prewhitening filter. These filters are popular because of their ease of implementation or particular performance measure that they optimize.

We now compare the performance of the optimum filter, $H_0(f)$, with the performance of the forenamed filters. The best way to make this comparison is by a numerical example. Consider the following:

$$G_S(f) = \begin{Bmatrix} .01 \exp(-f/750) & f < 5000 \\ 0 & f > 5000 \end{Bmatrix}$$

$$G_N(f) = \exp(-f/1000) \qquad \text{for } f > 0 \tag{11}$$

with integration time

$$T = 250 \text{ sec, and } P_F = 10^{-6}. \tag{12}$$

Equations (8) and (10) are used to determine performance. The results are presented in table 1; it shows that, for this example, $H_0(f)$ yields the best P_D.

Table 1. Filter Comparison

Filter	Power Transfer Function $\lvert H(f)\rvert^2$	Q	P_D
Optimum	Equation (9)	0.112199	0.71
Eckart	$G_S(f)/G_N^2(f)$	0.098704	0.58
Prewhitening	$1/G_N(f)$	0.096571	0.56
Ideal Passband	1	0.085089	0.44

If filtering is performed in the time domain, $H_0(f)$ is no more difficult to implement than the other filters. However, it is more difficult to design. The same holds true if filtering is performed in the frequency domain, except that the ideal passband filter is trivial to implement. The other advantage to the optimum filter is that the whole signal band need not be processed. The cutoff for the optimum filter in this example is 3455 Hz, whereas the other filters cut off at 5000 Hz.

UNKNOWN SPECTRA

When $G_S(f)$ and/or $G_N(f)$ are not completely known, then P_D is a function of the random variable Q. Two characteristics of P_D that are of interest are its expected value and variance.

Writing (10) as

$$P_D = F(Q), \tag{13}$$

and making use of section 7.7 of reference 2, we obtain the following approximations:

$$\begin{aligned} E[P_D] &\approx F(E[Q]) + F''(E[Q])\ \mathrm{Var}[Q]/2, \\ \mathrm{Var}[P_D] &\approx [F'(E[Q])]^2\ \mathrm{Var}[Q] \quad . \end{aligned} \tag{14}$$

Substituting (3) and (10) into (13), there follows

$$F'(E[Q]) = (2\pi)^{-1/2}\ T/2(E[Q]T)^{-1/2}\ \exp[-[(E[Q]T)^{1/2} + \Phi^{-1}(P_F)]^2/2],$$

$$F''(E[Q]) = -(2\pi)^{-1/2}\ T^2/4(E[Q]T)^{-1}\ [((E[Q]T)^{1/2} + \Phi^{-1}(P_F)) + (E[Q]T)^{-1/2}]$$

$$* \exp[-[(E[Q]T)^{1/2} + \Phi^{-1}(P_F)]^2/2]. \tag{15}$$

Appendix B is used to evaluate E[Q] and Var[Q] (see (B-8)).

Example

Consider the example from Appendix B with

$$T = 450 \text{ sec}, \; P_F = 10^{-7}, \; \Phi^{-1}(P_F) = -5.2 \quad . \tag{16}$$

Since $Q(R) = 0.08$, using (10), we obtain

$$P_D(R) = 0.7881 \quad . \tag{17}$$

Since $E[Q] = 0.08$ and $Var[Q] = 0.000256$, substituting into (14) and (15), we obtain

$$\begin{aligned} F'(E[Q]) &= 10.863433, \\ F''(E[Q]) &= -393.799455, \end{aligned} \tag{18}$$

and therefore

$$\begin{aligned} E[P_D] &= 0.7377, \\ Var\,[P_D] &= 0.0302. \end{aligned} \tag{19}$$

The error in the estimate of the signal spectrum has caused a decrease in expected performance.

CONCLUSION

The derivation for the broadband filter that maximizes probability of detection for a given false alarm rate has been provided. Performance improvement can be achieved using this filter. The detection performance will be degraded when the signal or noise spectrum must be estimated.

REFERENCES

1. A. H. Nuttall and A. F. Magaraci, Signal-to-Noise Ratios Required for Short-Term Narrowband Detection of Gaussian Processes, NUSC Technical Report 4417, 20 October 1972.

2. P. Meyer, Introductory Probability and Statistical Applications, Addison-Wesley Publishing Company, 1965.

APPENDIX A

DERIVATION OF OPTIMUM FILTER POWER TRANSFER FUNCTION

For notational convenience, we drop the f-dependence in (4) and (5) temporarily. Then defining

$$A = |H|^2 G_N, \quad R = G_S/G_N, \tag{A-1}$$

we have to maximize, by choice of A, the quantity

$$\left(\frac{S}{N}\right)^2 B = \frac{\left[\int AR\right]^4}{\left[\int A\right]^2 \left[\int A^2 R^2\right]} \quad . \tag{A-2}$$

Actually, we will consider the slightly more general form

$$Q \equiv \frac{\left[\int AR_1\right]^4}{\left[\int A\right]^2 \left[\int A^2 R_2\right]} \equiv \frac{\alpha^4}{\beta^2 \gamma} \quad , \tag{A-3}$$

where R_1 and R_2 are given known non-negative functions of frequency, f. The integrals in (A-3) are over the band W of interest, and would be written more accurately, for example, as

$$\alpha = \int AR_1 = \int_W df\ A(f) R_1(f) \quad . \tag{A-4}$$

Notice that the function A defined in (A-1) can never be negative.

To determine the optimum A, we let

$$A(f) = A_0(f) + \epsilon\eta(f) \quad , \tag{A-5}$$

where A_0 is the optimum, and η is an arbitrary perturbation. Substituting (A-5) into (A-3), we obtain

$$Q_0 + \delta Q = \frac{\left[\int (A_0 + \varepsilon\eta) R_1\right]^4}{\left[\int (A_0 + \varepsilon\eta)\right]^2 \left[\int (A_0 + \varepsilon\eta)^2 R_2\right]} \quad . \qquad \text{(A-6)}$$

Taking a partial of (A-6) with respect to ε, and then setting $\varepsilon = 0$, we get

$$\left[\beta_0^2 \gamma_0 \, 4 \, \alpha_0^3 \int \eta \, R_1 - \alpha_0^4 \left\{ 2\beta_0 \, \gamma_0 \int \eta + \beta_0^2 \, 2 \int A_0 \, \eta \, R_2 \right\}\right] \Big/ \left(\beta_0^4 \, \gamma_0^2\right) = 0 \text{ for any } \eta. \qquad \text{(A-7)}$$

Cancelling irrelevant constants, none of which can be zero, we find that

$$\int \eta \left\{ 2 \frac{\gamma_0}{\alpha_0} R_1 - \frac{\gamma_0}{\beta_0} - R_2 \, A_0 \right\} = 0 \quad \text{for any } \eta. \qquad \text{(A-8)}$$

The only way this can be is if the bracketed term is zero for all $f \varepsilon W$. Thus re-substituting the f dependence, the unique optimum A-function satisfies

$$R_2(f) \, A_0(f) = \frac{2\gamma_0}{\alpha_0} \left(R_1(f) - \frac{\alpha_0}{2\beta_0} \right) \quad \text{for } f \varepsilon W \quad . \qquad \text{(A-9)}$$

Now if $R_2(f)$ were zero in some subregion of W, where $R_1(f)$ is not zero, it is obvious from (A-3) that Q can be made equal to $+\infty$ by simply choosing A(f) to be non-zero only in that subregion, for then $\gamma=0$. Therefore, we exclude this case and require that $R_2(f) > 0$ for $f \varepsilon W$.

Also, since $A_0(f)$ can never be negative, we must replace any negative values that arise in (A-9) by zero. Accordingly we define region W_0 such that

$$R_1(f) \geq \frac{\alpha_0}{2\beta_0} \quad \text{for} \quad f \varepsilon W_0 \quad . \qquad \text{(A-10)}$$

Then the optimum A function is

$$A_0(f) = \left\{ \begin{array}{ll} \dfrac{2\gamma_0}{\alpha_0} \dfrac{1}{R_2(f)} \left[R_1(f) - \dfrac{\alpha_0}{2\beta_0} \right] & \text{for } f \varepsilon W_0 \\ \\ 0 \text{ for remaining } f \varepsilon W & \end{array} \right\} \qquad \text{(A-11)}$$

Several useful observations should be made at this point. First, from (A-3), the absolute scale of A or A_0 does not affect Q; so we will eventually be able to discard the scale factor $2\gamma_0/\alpha_0$ in (A-11). However, we must temporarily retain it in order to solve for the constants α_0, β_0, γ_0 defined in (A-3).

Second, we do not need γ_0 at all since it is merely a scale factor in (A-11); and we do not need to know α_0 and β_0 separately, but only their ratio. Also, α_0, β_0, γ_0 must all be positive since they are integrals of positive functions.

Finally, the form of the optimum $A_0(f)$ in (A-11) indicates that if $A_0(f)$ is zero, it is zero in the frequency regions where $R_1(f)$ is minimum, regardless of what $R_2(f)$ is. This is a very useful observation for determining W_0.

Now we substitute (A-11) in the individual terms of (A-3) to determine the constants α_0, β_0, γ_0. Dropping the zero subscripts on α, β, γ for notational convenience, we have

$$\alpha = \int A_0 R_1 = \frac{2\gamma}{\alpha} \int_{W_0} \frac{R_1}{R_2}\left(R_1 - \frac{\alpha}{2\beta}\right) = \frac{2\gamma}{\alpha}\left(I_2 - \frac{\alpha}{2\beta} I_1\right) ,$$

$$\beta = \int A_0 = \frac{2\gamma}{\alpha} \int_{W_0} \frac{1}{R_2}\left(R_1 - \frac{\alpha}{2\beta}\right) = \frac{2\gamma}{\alpha}\left(I_1 - \frac{\alpha}{2\beta} I_0\right) ,$$

$$\gamma = \int A_0^2 R_2 = \left(\frac{2\gamma}{\alpha}\right)^2 \int_{W_0} \frac{1}{R_2}\left(R_1 - \frac{\alpha}{2\beta}\right)^2 = \frac{4\gamma^2}{\alpha^2}\left(I_2 - \frac{\alpha}{\beta} I_1 + \frac{\alpha^2}{4\beta^2} I_0\right), \quad \text{(A-12)}$$

where

$$I_n \equiv \int_{W_0} \frac{R_1^n}{R_2} = \int_{W_0} df \, \frac{R_1^n(f)}{R_2(f)} . \quad \text{(A-13)}$$

Simplifying (A-12), we have the three simultaneous nonlinear equations

$$\alpha^2\beta = \gamma\left(2\beta\ I_2 - \alpha\ I_1\right) \quad ,$$

$$\alpha\beta^2 = \gamma\left(2\beta\ I_1 - \alpha\ I_0\right) \quad ,$$

$$\alpha^2\beta^2 = \gamma\left(4\beta^2\ I_2 - 4\alpha\beta\ I_1 + \alpha^2\ I_0\right) \quad . \tag{A-14}$$

At first sight, solution of this set looks formidable indeed. However, observe that the third equation is not independent of the first two; in fact, 2β times the first equation minus α times the second equation yields the third equation. So we discard the third equation in (A-14).

Next, eliminating the unwanted γ from the first two equations in (A-14), we obtain (restoring the zero subscripts on α, β, γ)

$$\left(\frac{\alpha_0}{\beta_0}\right)^2 I_0 - 3\,\frac{\alpha_0}{\beta_0}\, I_1 + 2\ I_2 = 0 \quad , \tag{A-15}$$

which is an equation in exactly the one unknown we fundamentally need in (A-11). The solutions are

$$\frac{\alpha_0}{\beta_0} = \frac{3\ I_1 \pm \left(9\ I_1^2 - 8\ I_0\ I_2\right)^{1/2}}{2\ I_0} \quad . \tag{A-16}$$

Define square root

$$S = \left(9\ I_1^2 - 8\ I_0\ I_2\right)^{1/2} \quad . \tag{A-17}$$

In order that a real solution for α_0/β_0 exist, as required by definitions (A-3), we must also have discriminant

$$D \equiv 9\ I_1^2 - 8\ I_0\ I_2 \geq 0 \quad . \tag{A-18}$$

Thus when we pick the region W_0 in (A-13), it must be done with constraint (A-18) in mind. If (A-18) is not satisfied, the corresponding choice of W_0 is disallowed.

It should be observed that (A-18) is a very tight restriction on I_0, I_1, I_2. For when we couple it with the Schwartz inequality (see (A-13))

$$I_1^2 = \left[\int \frac{R_1}{R_2}\right]^2 \leq \int \frac{1}{R_2} \int \frac{R_1^2}{R_2} = I_0 \; I_2 \quad ,$$

we have

$$\frac{8}{9} I_0 \; I_2 \leq I_1^2 \leq I_0 \; I_2 \quad .$$

In order to determine which of the two positive roots for α_0/β_0 is the correct one to retain in (A-16), we must substitute it into Q_0 and find which yields a maximum. From (A-16) and (A-17),

$$\left(\frac{\alpha_0}{\beta_0}\right)^2 = \left(\frac{3 \; I_1 \pm S}{2 \; I_0}\right)^2 = \frac{1}{4 \; I_0^2} \left(18 \; I_1^2 - 8 \; I_0 \; I_2 \pm 6 \; I_1 \; S\right) \quad , \qquad \text{(A-19)}$$

and from the first equation of (A-14),

$$\frac{\alpha_0^2}{\gamma_0} = 2 \; I_2 - \frac{\alpha_0}{\beta_0} I_1 = 2 \; I_2 - I_1 \left(\frac{3 \; I_1 \pm S}{2 \; I_0}\right) = \frac{1}{2 \; I_0} \left(4 \; I_0 \; I_2 - 3 \; I_1^2 \mp I_1 \; S\right) \quad . \qquad \text{(A-20)}$$

Then from (A-3), (A-19), and (A-20),

$$Q_0 = \frac{\alpha_0^2}{\beta_0^2} \frac{\alpha_0^2}{\gamma_0} = \frac{1}{2 \; I_0^3} \left(36 \; I_0 \; I_1^2 \; I_2 - 8 \; I_0^2 \; I_2^2 - 27 \; I_1^4 \mp I_1 \; S^3\right) \quad . \qquad \text{(A-21)}$$

This is obviously maximized by choosing the <u>lower</u> sign here and in (A-16). So we have

$$\frac{\alpha_0}{\beta_0} = \frac{3\, I_1 - S}{2\, I_0} \tag{A-22}$$

and

$$Q_0 = \frac{1}{2\, I_0^3} \left(36\, I_0\, I_1^2\, I_2 - 8\, I_0^2\, I_2^2 - 27\, I_1^4 + I_1\, S^3 \right)$$

$$= \frac{1}{2\, I_0^3} \left(27\, I_1^2 \left(I_0\, I_2 - I_1^2 \right) + I_0\, I_2\, D + I_1\, D\, S \right) \quad . \tag{A-23}$$

All the terms in (A-23) are obviously positive, the $I_0\, I_2 - I_1^2$ term following from Schwartz's inequality under (A-18).

We can now drop irrelevant scale factors in (A-11) and state that

$$A_0(f) = \frac{1}{R_2(f)} \left[R_1(f) - \frac{\alpha_0}{2\beta_0} \right]$$

$$= \frac{1}{R_2(f)} \left[R_1(f) - \frac{3\, I_1 - S}{4\, I_0} \right] \quad \text{for } f \in W_0 \quad . \tag{A-24}$$

The quantities I_0, I_1, I_2, and S needed in (A-22)-(A-24) are given by (A-13) and (A-17). We must keep in mind that (A-24) can never be negative and that (A-18) must be satisfied. The optimum filter power transfer function follows from (A-1) as

$$\left| H_0(f) \right|^2 = A_0(f) / G_N(f) \quad \text{for } f \in W_0 \quad . \tag{A-25}$$

If $\frac{\alpha_0}{2\beta_0}$ were known, we could easily determine W_0 by finding the frequencies where $R_1(f) = \frac{\alpha_0}{2\beta_0}$, as noted earlier under (A-11); but $\frac{\alpha_0}{2\beta_0}$ itself depends on W_0 through the quantities I_0, I_1, I_2. The way to determine $\frac{\alpha_0}{2\beta_0}$ is as follows. Hypothesize a value for $\frac{\alpha_0}{2\beta_0}$; call it c. Determine W_c from $R_1(f)$ by solving $R_1(f_c) = c$, and then choosing W_c as that region where $R_1(f) \geq c$; there may be several values of f_c. (There has been no need to consider $R_2(f)$ up to this point.) Compute

$$I_n = \int_{W_c} df \frac{R_1^n(f)}{R_2(f)} \text{ for } n = 0, 1, 2 \quad , \tag{A-26}$$

which depend on c. Compute D from (A-18); if $D < 0$, the hypothesized c value is disallowed. Otherwise, compute $S = \sqrt{D}$, and then α/β from (A-22). Now we will have the correct value of c when we get back $\alpha/(2\beta) = c$; so we have the identity

$$c = \frac{3\, I_1 - S}{4\, I_0} = \frac{3\, I_1 - \left(9\, I_1^2 - 8\, I_0\, I_2\right)^{1/2}}{4\, I_0} \tag{A-27}$$

or

$$2\, I_0\, c^2 - 3\, I_1\, c + I_2 = 0 \quad . \tag{A-28}$$

This last equation is identical to (A-15). Solution of (A-27) or (A-28) (numerically or analytically) determines $\alpha_0/(2\beta_0)$. Negative values for $A_0(f)$ are never encountered or contemplated via this approach. The only check to make is that D is not negative; if it is, the corresponding selection of c is disallowed and must be modified.

Example 1

We consider first the special case

$$R_1(f) = R(f), \quad R_2(f) = R^2(f), \tag{A-29}$$

where R(f) is a two-level function, i.e.,

$$R(f) = \begin{Bmatrix} r_1 & \text{for} & 0 < f < 1 \\ r_2 & \text{for} & 1 < f < 2 \end{Bmatrix} . \tag{A-30}$$

This example has equal bandwidths for each of the two frequency intervals, namely (0, 1) and (1, 2), normalized to 1 Hz for convenience. Without loss of generality, we let $r_1 \le r_2$ and define

$$r = \frac{r_2}{r_1} \quad ; \quad r \ge 1 \quad . \tag{A-31}$$

We now employ the procedure described in (A-26)-(A-28). First assume

$$c \le r_1 \quad . \tag{A-32}$$

Then $W_c = (0, 2)$, and from (A-13), (A-29), (A-30), and (A-18),

$$I_n = \int_0^2 df\, R^{n-2}(f) = r_1^{n-2} + r_2^{n-2} \equiv q_1^{2-n} + q_2^{2-n} \quad , \tag{A-33}$$

where $q_k \equiv 1/r_k$. Then

$$I_0 = q_1^2 + q_2^2,\ I_1 = q_1 + q_2,\ I_2 = 2, \quad S = \left(18\, q_1\, q_2 - 7\, q_1^2 - 7\, q_2^2\right)^{1/2} . \tag{A-34}$$

The identity (A-27) yields immediately

$$c = \frac{3\ q_1 + 3\ q_2 - \left(18\ q_1\ q_2 - 7\ q_1^2 - 7\ q_2^2\right)^{1/2}}{4\left(q_1^2 + q_2^2\right)} \equiv \frac{3\ q_1 + 3\ q_2 - S}{4\left(q_1^2 + q_2^2\right)} \tag{A-35}$$

for the value of $\frac{\alpha_0}{2\beta_0}$, provided only that the discriminant is non-negative. Satisfaction of this requirement yields (in conjunction with (A-31))

$$1 \leq r \leq \frac{9 + 4\sqrt{2}}{7} = 2.0938 \quad . \tag{A-36}$$

Also we have to confirm that (A-32) is true; i. e.,

$$\frac{3\ q_1 + 3\ q_2 - S}{4\left(q_1^2 + q_2^2\right)} \overset{?}{\leq} r_1 = \frac{1}{q_1} \quad . \tag{A-37}$$

This can be manipulated into the form

$$\frac{7}{4} q_2^2 + \left(\frac{3}{2} q_2 - q_1\right)^2 + q_1\ S \overset{?}{\geq} 0 \quad . \tag{A-38}$$

But since this is obviously always true, the only restriction, for this case of assumption (A-32), is (A-36). That is, (A-32) is an allowable assumption provided that (A-36) is satisfied.

We now consider the complementary case to (A-32); namely, we assume

$$c \geq r_1 \quad . \tag{A-39}$$

Then, $W_c = (1, 2)$, and

$$I_n = \int_1^2 df\ R^{n-2}(f) = r_2^{n-2} \quad ; \tag{A-40}$$

that is,

$$I_0 = q_2^2,\ I_1 = q_2,\ I_2 = 1,\ D = q_2^2 > 0,\ S = q_2 \quad . \tag{A-41}$$

Identity (A-27) yields immediately

$$c = \frac{3\ q_2 - q_2}{4\ q_2^2} = \frac{1}{2q_2} = \frac{r_2}{2} \tag{A-42}$$

for the value of $\alpha_0/(2\beta_0)$. In order to satisfy (A-39), we must have

$$\frac{r_2}{2} \geq r_1 \quad \text{or} \quad r \geq 2 \quad . \tag{A-43}$$

We now observe that conditions (A-36) and (A-43) cover all the possible values of r in $(1, +\infty)$; in fact, they overlap in the interval (2, 2.0938). In order to determine whether to select (A-35) or (A-42) in this overlap interval (and <u>both</u> are allowed), we must evaluate Q_0 in this interval and take the larger value.

We find that, from (A-23) and (A-34),

$$Q_0 = r_2^2 \left[\frac{13(1 + r^2)^2 + 36r(1 + r^2 - 3r) + (1 + r)\left(18r - 7 - 7\ r^2\right)^{3/2}}{2\left(1 + r^2\right)^3} \right] \tag{A-44}$$

for condition (A-36); while from (A-23) and (A-41),

$$Q_0 = r_2^2 \tag{A-45}$$

for condition (A-43). We also find that Q_0/r_2^2 in (A-44) decreases monotonically from the value 2 at r = 1, to the value 1 at

$$\hat{r} = 2.06353 \quad . \tag{A-46}$$

Thus, for $r \le \hat{r}$, the maximum Q_0 is given by (A-44), while for $r \ge \hat{r}$, the maximum Q_0 is given by (A-45). Q_0 is continuous with r, but has a discontinuous slope at $\hat{r}$.

As for the optimum A_0 function, we find, by use of (A-24) and (A-35), that when $1 \le r \le 2.0938$, with $q = (18r_1r_2 - 7r_1^2 - 7r_2^2)^{1/2}$,

$$A_0(f) = \left\{ \begin{array}{ll} \dfrac{4\,r_1^2 + r_2^2 - 3\,r_1\,r_2 + r_2\,q}{4\,r_1\left(r_1^2 + r_2^2\right)} & \text{for} \quad f \in (0,\,1) \\ \\ \dfrac{r_1^2 + 4\,r_2^2 - 3\,r_1\,r_2 + r_1\,q}{4\,r_2\left(r_1^2 + r_2^2\right)} & \text{for} \quad f \in (1,\,2) \end{array} \right\} . \tag{A-47}$$

Since the absolute scale of $A_0(f)$ is not important, only the ratio of filter gains is essential. It is, after considerable manipulation,

$$\frac{A_0^{(2)}(f)}{A_0^{(1)}(f)} = \frac{1 - r + \left(18r - 7 - 7r^2\right)^{1/2}}{2r(2 - r)} . \tag{A-48}$$

(When r = 2, this ratio is 3/2 via a limiting operation.) For r = 1, the ratio in (A-48) is 1 as expected, since the function R(f) in (A-30) is flat over (0, 2). At the other extreme, where r is the largest allowable value for this case (see (A-36)), the ratio of filter gains is

$$\frac{A_0^{(2)}(f)}{A_0^{(1)}(f)} = \frac{11 + 6\sqrt{2}}{7} = 2.7836 \quad \text{for} \quad r = \frac{9 + 4\sqrt{2}}{7} ; \tag{A-49}$$

while at $r = \hat{r} = 2.06353$, the ratio is 1.8440. Thus, ratio (A-48) increases monotonically from 1 (at r = 1) to the value in (A-49). Observe that this ratio is not $+\infty$ at this upper limit for r in this case; i.e., $A_0^{(1)}(f)$ is <u>not</u> zero at this value of $r = (9 + 4\sqrt{2})/7$.

For the complementary case, (A-43), we use (A-24) and (A-42) to get

$$A_0(f) = \begin{Bmatrix} 0 & \text{for} & f \in (0,\ 1) \\ \frac{1}{2r_2} & \text{for} & f \in (1,\ 2) \end{Bmatrix} . \qquad \text{(A-50)}$$

This is the optimum filter function for $r \geq \hat{r}$, while (A-47) is the optimum for $1 \leq r \leq \hat{r}$. So at $r = \hat{r}$, $A_0^{(1)}(f)$ jumps <u>abruptly</u> from a positive value to zero.

Example 2

Here we consider continuous functions of frequency

$$\left.\begin{aligned} R_1(f) &= \exp(-\tau f) \quad \text{for } f > 0 \\ R_2(f) &= \exp(-a\tau f) \quad \text{for } f > 0 \end{aligned}\right\} \quad W = (0, +\infty) \quad , \qquad \text{(A-51)}$$

where a is a dimensionless known constant; if a = 2, then $R_2(f) = R_1^2(f)$, the original case of interest in (A-2).

We use the procedure described in (A-26)-(A-28). Let c=1. Then $W_c = (0,\ f_c)$, where f_c is the solution of

$$R_1(f_c) = \exp\ (-\tau f_c) = c; \quad \tau f_c = -\ln(c) \quad . \qquad \text{(A-52)}$$

Then,

$$I_n = \int_{W_c} df\ \frac{R_1^n(f)}{R_2(f)} = \int_0^{f_c} df\ \exp\ (-n\tau f + a\tau f) = \frac{\exp[(a-n)\tau f_c] - 1}{(a-n)\tau} \quad . \qquad \text{(A-53)}$$

The only case we will consider here is $a = 2$; then, using (A-52),

$$I_0 = \frac{1}{\tau}\frac{1-c^2}{2c^2}, \quad I_1 = \frac{1}{\tau}\frac{1-c}{c}, \quad I_2 = f_c = -\frac{1}{\tau}\ln(c) \quad . \tag{A-54}$$

Identity (A-28) yields

$$1 - c^2 - 3\,(1 - c) - \ln(c) = 0 \quad ,$$

$$(c - 1)\,(c - 2) + \ln(c) = 0 \quad . \tag{A-55}$$

There are only two solutions of this equation; they are $c = 1$ (disallowed) and

$$c = .316197 = \frac{\alpha_0}{2\beta_0} \quad . \tag{A-56}$$

Then (A-52) yields

$$\tau f_c = 1.15139 \quad . \tag{A-57}$$

The value of Q_0 is obtained by substituting (A-54) and (A-56) in (A-23), yielding

$$\tau Q_0 = .373997 \quad . \tag{A-58}$$

The optimum filter follows upon use of (A-51), (A-54), and (A-56) in (A-24); i.e.,

$$A_0(f) = \frac{c^u - c}{c^{2u}} \quad \text{for} \quad 0 < u < 1 \tag{A-59}$$

where

$$u = \frac{f}{f_c} \quad . \tag{A-60}$$

A plot of (A-59) is given in figure A-1. The optimum $A_0(f)$ goes to zero at $f = f_c$; it also lends heaviest emphasis to frequencies near $.4f_c$, rather than at zero where $R_1(f)$ and $R_2(f)$ are largest.

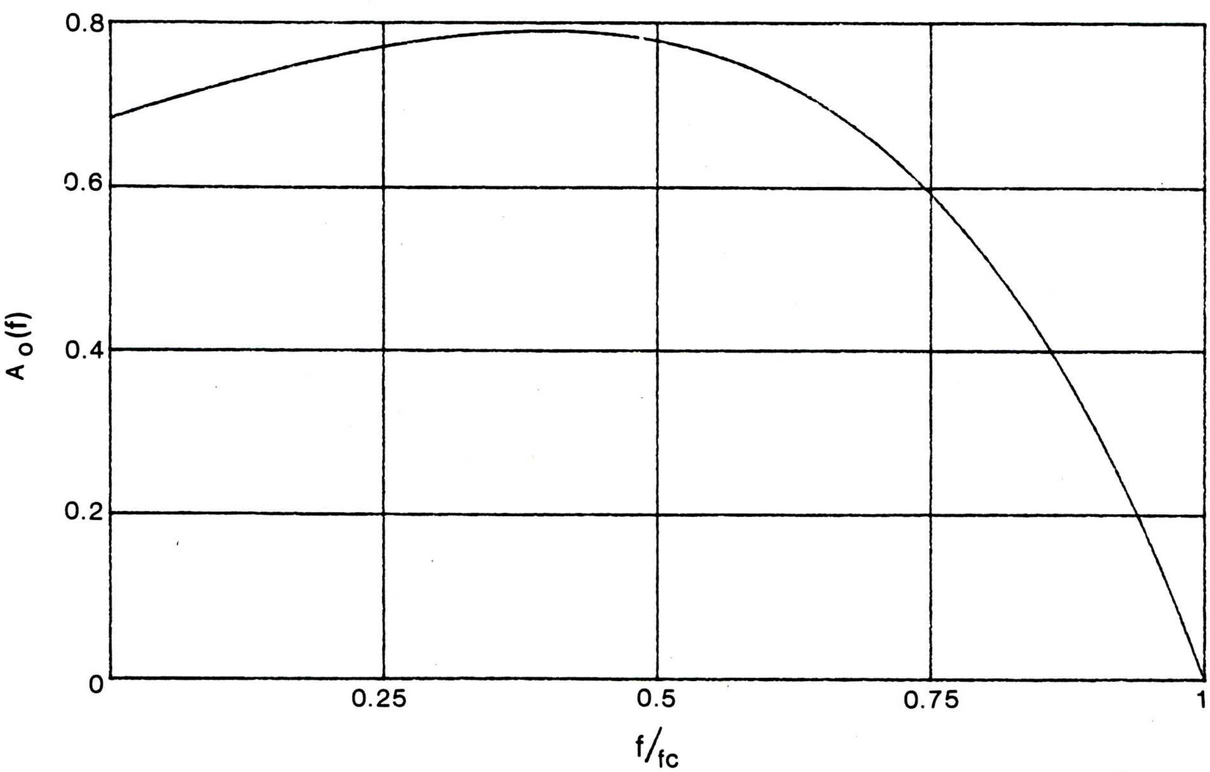

Figure A-1. Optimum $A_0(f)$ of (A-59)

APPENDIX B

SENSITIVITY ANALYSIS OF FILTER

It often happens that G_S and/or G_N are not completely known but must be "guessed" or estimated. Furthermore H(f) usually cannot be realized exactly. All of these factors influence performance and a derived optimum filter may not be optimum in fact.

When $\tilde{R}$ (the estimate of R) differs from the true R in a deterministic manner, Q can be computed via (B-1). However, if $\tilde{R}$ differs from R in a probabilistic manner, then Q is a random variable and one can only compute the expected performance.

From (A-1) and (A-3) in appendix A,

$$Q = \frac{\left[\int A\tilde{R}\right]^4}{\left[\int A\right]^2 \left[\int A^2\tilde{R}^2\right]} = \frac{\alpha^4}{\beta^2\gamma} \quad , \qquad \text{(B-1)}$$

where $\tilde{R} = G_S/G_N$, $A = |H|^2 G_N$.

The subsequent analysis assumes that only G_S must be estimated (i.e., G_N is exactly known and H can be exactly implemented). The analysis can be extended to the case of implementation errors in the filter and unknown G_N.

We assume that G_S is estimated by breaking frequency region W into n equilength subregions W_i, $1 \le i \le n$, where the power in each subregion is measured. Thus, n must be chosen large enough so that G_S and G_N remain fairly constant within each subregion. The problem is then transformed into the discrete domain. Namely, we can now rewrite (B-1) as

$$Q(\underline{\tilde{R}}) = \frac{\left[\sum_{i=1}^{n} A_i\tilde{R}_i\right]^4}{\left[\sum_{i=1}^{n} A_i\right]^2 \left[\sum_{i=1}^{n} A_i^2\,\tilde{R}_i^2\right]} = \frac{\alpha^4}{\beta^2\gamma} \quad , \qquad \text{(B-2)}$$

where Q is a function of the random variables $\tilde{R}_i$, $1 \leq i \leq n$.

Q can be expressed as a multidimensional Taylor series, i.e.,

$$Q(R) = Q(\tilde{\underline{R}}) + \sum_{i=1}^{n} \partial Q/\partial R_i \Big|_{\tilde{\underline{R}}} (R_i - \tilde{R}_i)$$

$$+ \frac{1}{2!} \sum_{i=1}^{n} \sum_{j=1}^{n} \partial^2 Q/\partial R_i \partial R_j \Big|_{\tilde{\underline{R}}} (R_i - \tilde{R}_i)(R_j - \tilde{R}_j)$$

$$+ \ldots \tag{B-3}$$

If $(R_i - \tilde{R}_i)$ is small, then E[Q] and Var[Q] may be approximated by neglecting high order terms in the Taylor series. Furthermore, if

$$E[R_i - \tilde{R}_i] = 0,$$

$$\mathrm{Var}[\tilde{R}_i] = \sigma_i^2, \tag{B-4}$$

and all the $\{\tilde{R}_i\}$ are independent, then

$$E[Q] \approx Q(\tilde{\underline{R}}) + \frac{1}{2} \sum_{i=1}^{n} \partial^2 Q/\partial R_i{}^2 \Big|_{\tilde{\underline{R}}} \sigma_i^2 ,$$

$$\mathrm{Var}[Q] \approx \sum_{i=1}^{n} (\partial Q/\partial R_i)^2 \Big|_{\tilde{\underline{R}}} \sigma_i^2 . \tag{B-5}$$

The chain rule can now be used to evaluate $\partial Q/\partial R_i$ and $\partial^2 Q/\partial R_i^2$. Making use of (B-2), we find that

$$\partial Q/\partial R_i \Big|_{\tilde{\underline{R}}} = \frac{1}{\beta^2} \left[\frac{4A_i \alpha^3}{\gamma} - \frac{2A_i^2 R_i \alpha^4}{\gamma^2} \right]_{\tilde{\underline{R}}} ,$$

$$\partial^2 Q/\partial R_i^2\Big|_{\underset{\sim}{\tilde{R}}} = \frac{1}{\beta^2}\left[4A_i\left(\frac{3A_i\alpha^2}{\gamma} - \frac{2A_i^2 R_i \alpha^3}{\gamma^2}\right)\right.$$

$$\left. + 2A_i^2\left(\frac{4A_i^2 R_i^2 \alpha^4}{\gamma^3} - \frac{4A_i R_i \alpha^3 + \alpha^4}{\gamma^2}\right)\right]_{\underset{\sim}{\tilde{R}}} \quad . \quad \text{(B-6)}$$

Rearranging (B-6), we obtain

$$\partial Q/\partial R_i\Big|_{\underset{\sim}{\tilde{R}}} = Q(\tilde{R})\left[4A_i/\alpha - 2A_i^2 R_i/\gamma\right]_{\underset{\sim}{\tilde{R}}} \quad ,$$

$$\partial^2 Q/\partial R_i^2\Big|_{\underset{\sim}{\tilde{R}}} = Q(\tilde{R})\left[12A_i^2/\alpha^2 - 16A_i^3 R_i/(\alpha\gamma)\right.$$

$$\left. + 8A_i^4 R_i^2/\gamma^2 - 2A_i^2/\gamma\right]_{\underset{\sim}{\tilde{R}}} \quad . \quad \text{(B-7)}$$

Substituting (B-7) into (B-5), we obtain

$$E[Q] = Q(\underset{\sim}{\tilde{R}})\left[1 + \frac{1}{2}\sum_{i=1}^{n}(12A_i^2/\alpha^2 - 16A_i^3 R_i/(\alpha\gamma)\right.$$

$$\left. + 8A_i^4 R_i^2/\gamma^2 - 2A_i^2/\gamma)\,\sigma_i^2\right]_{\underset{\sim}{\tilde{R}}} \quad ,$$

$$\mathrm{Var}[Q] = Q^2(\underset{\sim}{\tilde{R}})\left[\sum_{i=1}^{n}(4A_i/\alpha - 2A_i^2 R_i/\gamma)^2\right]_{\underset{\sim}{\tilde{R}}} \sigma_i^2 \quad . \quad \text{(B-8)}$$

<u>Example</u>

Reconsider Ex. 1 from Appendix A with

$$\begin{aligned} &E\,[G_{S1}] = 2 \quad , \\ &\mathrm{Var}\,[G_{S1}] = 0.04 \quad , \\ &E\,[G_{S2}] = 2 \quad , \\ &\mathrm{Var}\,[G_{S2}] = 0.12 \quad , \\ &G_N = 10. \end{aligned} \qquad \text{(B-9)}$$

Then,

$$\begin{aligned} R_1 &= 0.2 \quad , \\ R_2 &= 0.2 \quad , \\ \sigma_1^2 &= 0.04/G_N^2 = 0.0004 \quad , \\ \sigma_2^2 &= 0.12/G_N^2 = 0.0012 \quad . \end{aligned} \tag{B-10}$$

For this trivial example, by inspection let

$$H_1^2 = 1 \quad ,$$

$$H_2^2 = 1 \quad . \tag{B-11}$$

Then, from (B-2)

$$\begin{aligned} A_1 &= 10 \quad , \\ A_2 &= 10 \quad , \end{aligned} \tag{B-12}$$

and at $\underset{\sim}{\tilde{R}}$

$$\begin{aligned} \alpha &= 4 \quad , \\ \beta &= 20 \quad , \\ \gamma &= 8 \quad , \\ Q &= 0.08. \end{aligned} \tag{B-13}$$

Substituting (B-10), (B-12), and (B-13) into (B-8)

$$\begin{aligned} E[Q] &\approx 0.08 \quad , \\ \text{Var}\,[Q] &\approx 0.000256, \\ \text{Std dev}\,[Q] &\approx 0.016 \quad . \end{aligned} \tag{B-14}$$

So Q lies most often in the region 0.08 ± 0.016.

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